Number of Diagonals of a Convex Polygon

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David Simpson postulated (c. 1975–76) that the number of diagonals d that can be drawn in a convex polygon of n sides is given by the expression

$$d = (n-2) + (n-3) + (n-4) + \cdots + 2. \tag{1}$$

Around the same time, David Benning conjectured the simpler formula

$$d = \frac{n(n-3)}{2}. \tag{2}$$

Equations (1) and (2) can be shown to be equivalent. First, we write Simpson's formula (Eq. 1) in the more compact form

$$d = \sum_{k=2}^{n-2} (n-k). \tag{3}$$

Now apply some summation algebra:

$$\begin{aligned}
d &= \left[\sum_{k=2}^{n-2} (n-k) \right] + (n-1) - (n-1) && (4)\\
&= \left[\sum_{k=1}^{n-2} (n-k) \right] - (n-1) && (5)\\
&= \left[\sum_{k=1}^{n-2} n \right] - \left[\sum_{k=1}^{n-2} k \right] - (n-1) && (6)\\
&= [n(n-2)] - \left[\frac{(n-2)(n-1)}{2} \right] - (n-1) && (7)\\
&= \frac{2(n^2-2n)}{2} - \left[\frac{(n-2)(n-1)}{2} \right] - \frac{2(n-1)}{2} && (8)\\
&= \frac{2n^2 - 4n - n^2 + 3n - 2 - 2n + 2}{2} && (9)\\
&= \frac{n^2-3n}{2} && (10)\\
&= \frac{n(n-3)}{2} && (11)
\end{aligned}$$

which is Benning's formula.

Benning's formula can be arrived at simply. From each of the n vertices, we can draw $n-1$ lines to the $n-1$ other vertices. But the two lines drawn to the two *adjacent* vertices form *sides* of the polygon, not diagonals. Therefore only $(n-1)-2=n-3$ diagonals can be drawn from each of the n vertices, for a total of $n(n-3)$. But each diagonal has been counted twice, so the total number of unique diagonals is $n(n-3)/2$, *Q.E.D.*