

Quantum Harmonic Oscillator Ladder Operators

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Here a is the demotion (annihilation, lowering) operator; and a^\dagger is the promotion (creation, raising) operator for the quantum-mechanical simple harmonic oscillator.

One Operator

$$\begin{aligned} a |n\rangle &= \sqrt{n} |n-1\rangle \\ a^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \end{aligned}$$

Two Operators

$$\begin{aligned} aa |n\rangle &= \sqrt{n(n-1)} |n-2\rangle \\ aa^\dagger |n\rangle &= (n+1) |n\rangle \\ a^\dagger a |n\rangle &= n |n\rangle \\ a^\dagger a^\dagger |n\rangle &= \sqrt{(n+1)(n+2)} |n+2\rangle \end{aligned}$$

Three Operators

$$\begin{aligned} aaa |n\rangle &= \sqrt{n(n-1)(n-2)} |n-3\rangle \\ aaa^\dagger |n\rangle &= (n+1)\sqrt{n} |n-1\rangle \\ aa^\dagger a |n\rangle &= n^{3/2} |n-1\rangle \\ aa^\dagger a^\dagger |n\rangle &= (n+2)\sqrt{n+1} |n+1\rangle \\ a^\dagger aa |n\rangle &= (n-1)\sqrt{n} |n-1\rangle \\ a^\dagger aa^\dagger |n\rangle &= (n+1)^{3/2} |n+1\rangle \\ a^\dagger a^\dagger a |n\rangle &= n\sqrt{n+1} |n+1\rangle \\ a^\dagger a^\dagger a^\dagger |n\rangle &= \sqrt{(n+1)(n+2)(n+3)} |n+3\rangle \end{aligned}$$

Four Operators

$$\begin{aligned}
aaaa |n\rangle &= \sqrt{n(n-1)(n-2)(n-3)} |n-4\rangle \\
aaaa^\dagger |n\rangle &= (n+1)\sqrt{n(n-1)} |n-2\rangle \\
aaa^\dagger a |n\rangle &= n^{3/2} \sqrt{n-1} |n-2\rangle \\
aaa^\dagger a^\dagger |n\rangle &= (n+1)(n+2) |n\rangle \\
aa^\dagger aa |n\rangle &= (n-1)^{3/2} \sqrt{n} |n-2\rangle \\
aa^\dagger aa^\dagger |n\rangle &= (n+1)^2 |n\rangle \\
aa^\dagger a^\dagger a |n\rangle &= n(n+1) |n\rangle \\
aa^\dagger a^\dagger a^\dagger |n\rangle &= (n+3)\sqrt{(n+1)(n+2)} |n+2\rangle \\
a^\dagger aaa |n\rangle &= (n-2)\sqrt{n(n-1)} |n-2\rangle \\
a^\dagger aaa^\dagger |n\rangle &= n(n+1) |n\rangle \\
a^\dagger aa^\dagger a |n\rangle &= n^2 |n\rangle \\
a^\dagger aa^\dagger a^\dagger |n\rangle &= (n+2)^{3/2} \sqrt{n+1} |n+2\rangle \\
a^\dagger a^\dagger aa |n\rangle &= n(n-1) |n\rangle \\
a^\dagger a^\dagger aa^\dagger |n\rangle &= (n+1)^{3/2} \sqrt{n+2} |n+2\rangle \\
a^\dagger a^\dagger a^\dagger a |n\rangle &= n\sqrt{(n+1)(n+2)} |n+2\rangle \\
a^\dagger a^\dagger a^\dagger a^\dagger |n\rangle &= \sqrt{(n+1)(n+2)(n+3)(n+4)} |n+4\rangle
\end{aligned}$$

Five Operators

$$\begin{aligned}
aaaaaa |n\rangle &= \sqrt{n(n-1)(n-2)(n-3)(n-4)} |n-5\rangle \\
aaaaaa^\dagger |n\rangle &= (n+1)\sqrt{n(n-1)(n-2)} |n-3\rangle \\
aaaaa^\dagger a |n\rangle &= n^{3/2} \sqrt{(n-1)(n-2)} |n-3\rangle \\
aaaaa^\dagger a^\dagger |n\rangle &= (n+1)(n+2)\sqrt{n} |n-1\rangle \\
aaa^\dagger aa |n\rangle &= (n-1)^{3/2} \sqrt{n(n-2)} |n-3\rangle \\
aaa^\dagger aa^\dagger |n\rangle &= (n+1)^2 \sqrt{n} |n-1\rangle \\
aaa^\dagger a^\dagger a |n\rangle &= (n+1)n^{3/2} |n-1\rangle \\
aaa^\dagger a^\dagger a^\dagger |n\rangle &= (n+2)(n+3)\sqrt{n+1} |n+1\rangle \\
aa^\dagger aaa |n\rangle &= (n-2)^{3/2} \sqrt{n(n-1)} |n-3\rangle \\
aa^\dagger aaa^\dagger |n\rangle &= (n+1)n^{3/2} |n-1\rangle \\
aa^\dagger aa^\dagger a |n\rangle &= n^{5/2} |n\rangle \\
aa^\dagger aa^\dagger a^\dagger |n\rangle &= (n+2)^2 \sqrt{n+1} |n+1\rangle \\
aa^\dagger a^\dagger aa |n\rangle &= (n-1)n^{3/2} |n-1\rangle \\
aa^\dagger a^\dagger aa^\dagger |n\rangle &= (n+2)(n+1)^{3/2} |n+1\rangle \\
aa^\dagger a^\dagger a^\dagger a |n\rangle &= n(n+2)\sqrt{(n+1)} |n+1\rangle \\
aa^\dagger a^\dagger a^\dagger a^\dagger |n\rangle &= (n+4)\sqrt{(n+1)(n+2)(n+3)} |n+3\rangle \\
a^\dagger aaaa |n\rangle &= (n-3)\sqrt{n(n-1)(n-2)} |n-3\rangle \\
a^\dagger aaaa^\dagger |n\rangle &= (n-1)(n+1)\sqrt{n} |n-1\rangle \\
a^\dagger aaa^\dagger a |n\rangle &= (n-1)n^{3/2} |n-1\rangle \\
a^\dagger aaa^\dagger a^\dagger |n\rangle &= (n+2)(n+1)^{3/2} |n+1\rangle \\
a^\dagger aa^\dagger aa |n\rangle &= (n-1)^2 \sqrt{n} |n-1\rangle \\
a^\dagger aa^\dagger aa^\dagger |n\rangle &= (n+1)^{5/2} |n+1\rangle \\
a^\dagger aa^\dagger a^\dagger a |n\rangle &= n(n+1)^{3/2} |n+1\rangle \\
a^\dagger aa^\dagger a^\dagger a^\dagger |n\rangle &= (n+3)^{3/2} \sqrt{(n+1)(n+2)} |n+3\rangle \\
a^\dagger a^\dagger aaa |n\rangle &= (n-1)(n-2)\sqrt{n} |n-1\rangle \\
a^\dagger a^\dagger aaa^\dagger |n\rangle &= n(n+1)^{3/2} |n+1\rangle \\
a^\dagger a^\dagger aa^\dagger a |n\rangle &= n^2 \sqrt{n+1} |n+1\rangle \\
a^\dagger a^\dagger aa^\dagger a^\dagger |n\rangle &= (n+2)^{3/2} \sqrt{(n+1)(n+3)} |n+3\rangle \\
a^\dagger a^\dagger a^\dagger aa |n\rangle &= n(n-1)\sqrt{n+1} |n+1\rangle \\
a^\dagger a^\dagger a^\dagger aa^\dagger |n\rangle &= (n+1)^{3/2} \sqrt{(n+2)(n+3)} |n+3\rangle \\
a^\dagger a^\dagger a^\dagger a^\dagger a |n\rangle &= n\sqrt{(n+1)(n+2)(n+3)} |n+3\rangle \\
a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger |n\rangle &= \sqrt{(n+1)(n+2)(n+3)(n+4)(n+5)} |n+5\rangle
\end{aligned}$$