Power Series

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Chapter 1

Operations With Series

We will define series

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots , \tag{1.1}$$

$$\sum_{k=0}^{\infty} b_k x^k = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots , \tag{1.2}$$

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots . \tag{1.3}$$

The first two series will be assumed to be the known operands, so that the coefficients a_k and b_k are known. We will wish to find the coefficients c_k.

1.1 Sum

The sum of two series is given by

$$\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} c_k x^k. \tag{1.4}$$

where

$$\begin{aligned}
c_0 &= a_0 + b_0 && (1.5)\\
c_1 &= a_1 + b_1 && (1.6)\\
c_2 &= a_2 + b_2 && (1.7)\\
c_3 &= a_3 + b_3 && (1.8)\\
c_4 &= a_4 + b_4 && (1.9)\\
c_5 &= a_5 + b_5 && (1.10)\\
c_6 &= a_6 + b_6 && (1.11)\\
c_7 &= a_7 + b_7 && (1.12)\\
c_8 &= a_8 + b_8 && (1.13)
\end{aligned}$$

In general,

$$c_k = a_k + b_k \tag{1.14}$$

1.2 Difference

The difference of two series is given by

$$\sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} c_k x^k. \tag{1.15}$$

where

$$\begin{aligned}
c_0 &= a_0 - b_0 && (1.16)\\
c_1 &= a_1 - b_1 && (1.17)\\
c_2 &= a_2 - b_2 && (1.18)\\
c_3 &= a_3 - b_3 && (1.19)\\
c_4 &= a_4 - b_4 && (1.20)\\
c_5 &= a_5 - b_5 && (1.21)\\
c_6 &= a_6 - b_6 && (1.22)\\
c_7 &= a_7 - b_7 && (1.23)\\
c_8 &= a_8 - b_8 && (1.24)
\end{aligned}$$

In general,

$$c_k = a_k - b_k \tag{1.25}$$

1.3 Product

The product of two series is given by

$$\left(\sum_{k=0}^{\infty} a_k x^k\right) \times \left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{k=0}^{\infty} c_k x^k \tag{1.26}$$

where

$$c_0 = a_0 b_0 \tag{1.27}$$
$$c_1 = a_0 b_1 + a_1 b_0 \tag{1.28}$$
$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 \tag{1.29}$$
$$c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \tag{1.30}$$
$$c_4 = a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0 \tag{1.31}$$
$$c_5 = a_0 b_5 + a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 + a_5 b_0 \tag{1.32}$$
$$c_6 = a_0 b_6 + a_1 b_5 + a_2 b_4 + a_3 b_3 + a_4 b_2 + a_5 b_1 + a_6 b_0 \tag{1.33}$$
$$c_7 = a_0 b_7 + a_1 b_6 + a_2 b_5 + a_3 b_4 + a_4 b_3 + a_5 b_2 + a_6 b_1 + a_7 b_0 \tag{1.34}$$
$$c_8 = a_0 b_8 + a_1 b_7 + a_2 b_6 + a_3 b_5 + a_4 b_4 + a_5 b_3 + a_6 b_2 + a_7 b_1 + a_8 b_0 \tag{1.35}$$

In general,

$$c_k = \sum_{i=0}^{k} a_i b_{k-i} \tag{1.36}$$

1.4 Quotient

The quotient of two series is given by

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \sum_{k=0}^{\infty} c_k x^k \tag{1.37}$$

where

$$\begin{aligned}
c_0 &= \frac{b_0}{a_0} && (1.38)\\
c_1 &= \frac{1}{a_0}\Big[b_1 - c_0 a_1\Big] && (1.39)\\
c_2 &= \frac{1}{a_0}\Big[b_2 - c_1 a_1 - c_0 a_2\Big] && (1.40)\\
c_3 &= \frac{1}{a_0}\Big[b_3 - c_2 a_1 - c_1 a_2 - c_0 a_3\Big] && (1.41)\\
c_4 &= \frac{1}{a_0}\Big[b_4 - c_3 a_1 - c_2 a_2 - c_1 a_3 - c_0 a_4\Big] && (1.42)\\
c_5 &= \frac{1}{a_0}\Big[b_5 - c_4 a_1 - c_3 a_2 - c_2 a_3 - c_1 a_4 - c_0 a_5\Big] && (1.43)\\
c_6 &= \frac{1}{a_0}\Big[b_6 - c_5 a_1 - c_4 a_2 - c_3 a_3 - c_2 a_4 - c_1 a_5 - c_0 a_6\Big] && (1.44)\\
c_7 &= \frac{1}{a_0}\Big[b_7 - c_6 a_1 - c_5 a_2 - c_4 a_3 - c_3 a_4 - c_2 a_5 - c_1 a_6 - c_0 a_7\Big] && (1.45)\\
c_8 &= \frac{1}{a_0}\Big[b_8 - c_7 a_1 - c_6 a_2 - c_5 a_3 - c_4 a_4 - c_3 a_5 - c_2 a_6 - c_1 a_7 - c_0 a_8\Big] && (1.46)
\end{aligned}$$

Note that we must have $a_0 \neq 0$. If dividing by a series for which $a_0 = 0$, it will be necessary to factor the appropriate power of x from the divisor series to get $a_0 = 0$ before applying these formulae.

In general, for $k > 0$,

$$c_k = \frac{1}{a_0}\left[b_k - \sum_{i=1}^{k} c_{k-i} a_i\right] = \frac{1}{a_0}\left[b_k - \sum_{i=0}^{k-1} a_{k-i} c_i\right] \tag{1.47}$$

In terms of a_k only, for $k > 0$, the coefficient c_k may be found from the determinant of a $k \times k$ matrix:

$$c_k = \frac{(-1)^k}{a_0^{k+1}} \begin{vmatrix} (a_1 b_0 - a_0 b_1) & a_0 & 0 & \dots & 0 \\ (a_2 b_0 - a_0 b_2) & a_1 & a_0 & \dots & 0 \\ (a_3 b_0 - a_0 b_3) & a_2 & a_1 & \dots & 0 \\ \dots & & & & \\ \dots & & & & \\ (a_{k-1} b_0 - a_0 b_{k-1}) & a_{k-2} & a_{k-3} & \dots & a_0 \\ (a_k b_0 - a_0 b_k) & a_{k-1} & a_{k-2} & \dots & a_1 \end{vmatrix} \tag{1.48}$$

1.5 Reciprocal

The reciprocal of a series may be found from the result of the previous section, setting $b_0 = 1$ and $b_k = 0$ for $k > 0$. The result is

$$\frac{1}{\sum_{k=0}^{\infty} a_k x^k} = \sum_{k=0}^{\infty} c_k x^k \tag{1.49}$$

where

$$c_0 = \frac{1}{a_0} \tag{1.50}$$

$$c_1 = \frac{-c_0 a_1}{a_0} \tag{1.51}$$

$$c_2 = \frac{1}{a_0}\Big[-c_1 a_1 - c_0 a_2\Big] \tag{1.52}$$

$$c_3 = \frac{1}{a_0}\Big[-c_2 a_1 - c_1 a_2 - c_0 a_3\Big] \tag{1.53}$$

$$c_4 = \frac{1}{a_0}\Big[-c_3 a_1 - c_2 a_2 - c_1 a_3 - c_0 a_4\Big] \tag{1.54}$$

$$c_5 = \frac{1}{a_0}\Big[-c_4 a_1 - c_3 a_2 - c_2 a_3 - c_1 a_4 - c_0 a_5\Big] \tag{1.55}$$

$$c_6 = \frac{1}{a_0}\Big[-c_5 a_1 - c_4 a_2 - c_3 a_3 - c_2 a_4 - c_1 a_5 - c_0 a_6\Big] \tag{1.56}$$

$$c_7 = \frac{1}{a_0}\Big[-c_6 a_1 - c_5 a_2 - c_4 a_3 - c_3 a_4 - c_2 a_5 - c_1 a_6 - c_0 a_7\Big] \tag{1.57}$$

$$c_8 = \frac{1}{a_0}\Big[-c_7 a_1 - c_6 a_2 - c_5 a_3 - c_4 a_4 - c_3 a_5 - c_2 a_6 - c_1 a_7 - c_0 a_8\Big] \tag{1.58}$$

Note that we must have $a_0 \neq 0$. If dividing by a series for which $a_0 = 0$, it will be necessary to factor the appropriate power of x from the divisor series to get $a_0 = 0$ before applying these formulae.

In general, for $k > 0$,

$$c_k = -\frac{1}{a_0} \sum_{i=1}^{k} c_{k-i} a_i = -\frac{1}{a_0} \sum_{i=0}^{k-1} a_{k-i} c_i \tag{1.59}$$

This latter result may also be found by setting $n = -1$ into the "powers" formula.

In terms of a_k only, for $k > 0$, the coefficient c_k will be found from the determinant of a $k \times k$ matrix:

$$c_k = \frac{(-1)^k}{a_0^{k+1}} \begin{vmatrix} a_1 & a_0 & 0 & \dots & 0 \\ a_2 & a_1 & a_0 & \dots & 0 \\ a_3 & a_2 & a_1 & \dots & 0 \\ \dots & & & & \\ \dots & & & & \\ a_{k-1} & a_{k-2} & a_{k-3} & \dots & a_0 \\ a_k & a_{k-1} & a_{k-2} & \dots & a_1 \end{vmatrix} \tag{1.60}$$

1.6 Powers

A series may be taken to a power:

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^n = \sum_{k=0}^{\infty} c_k x^k \tag{1.61}$$

Here n may be positive or negative, integer or fractional. The coefficients c_k are

$$c_0 = a_0^n \tag{1.62}$$

$$c_1 = \frac{n a_1 c_0}{a_0} \tag{1.63}$$

$$c_2 = \frac{1}{2a_0}\left[(n-1)a_1c_1 + 2na_2c_0\right] \tag{1.64}$$

$$c_3 = \frac{1}{3a_0}\left[(n-2)a_1c_2 + (2n-1)a_2c_1 + 3na_3c_0\right] \tag{1.65}$$

$$c_4 = \frac{1}{4a_0}\left[(n-3)a_1c_3 + (2n-2)a_2c_2 + (3n-1)a_3c_1 + 4na_4c_0\right] \tag{1.66}$$

$$c_5 = \frac{1}{5a_0}\left[(n-4)a_1c_4 + (2n-3)a_2c_3 + (3n-2)a_3c_2 + (4n-1)a_4c_1 + 5na_5c_0\right] \tag{1.67}$$

$$\begin{aligned} c_6 = \frac{1}{6a_0}\Big[&(n-5)a_1c_5 + (2n-4)a_2c_4 + (3n-3)a_3c_3 + (4n-2)a_4c_2 \\ &+ (5n-1)a_5c_1 + 6na_6c_0\Big] \end{aligned} \tag{1.68}$$

$$\begin{aligned} c_7 = \frac{1}{7a_0}\Big[&(n-6)a_1c_6 + (2n-5)a_2c_5 + (3n-4)a_3c_4 + (4n-3)a_4c_3 \\ &+ (5n-2)a_5c_2 + (6n-1)a_6c_1 + 7na_7c_0\Big] \end{aligned} \tag{1.69}$$

$$\begin{aligned} c_8 = \frac{1}{8a_0}\Big[&(n-7)a_1c_7 + (2n-6)a_2c_6 + (3n-5)a_3c_5 + (4n-4)a_4c_4 \\ &+ (5n-3)a_5c_3 + (6n-2)a_6c_2 + (7n-1)a_7c_1 + 8na_8c_0\Big] \end{aligned} \tag{1.70}$$

In general, for $k > 0$,

$$c_k = \frac{1}{ka_0}\sum_{i=0}^{k-1}[(k-i)n - i]a_{k-i}c_i \tag{1.71}$$

1.7 Square

The square of a series is found by substituting $n = 2$ into this result:

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^2 = \sum_{k=0}^{\infty} c_k x^k \tag{1.72}$$

The coefficients c_k are

$$c_0 = a_0^2 \tag{1.73}$$

$$c_1 = \frac{2a_1c_0}{a_0} \tag{1.74}$$

$$c_2 = \frac{1}{2a_0}\left[a_1c_1 + 4a_2c_0\right] \tag{1.75}$$

$$c_3 = \frac{1}{a_0}\left[a_2c_1 + 2a_3c_0\right] \tag{1.76}$$

$$c_4 = \frac{1}{4a_0}\left[-a_1c_3 + 2a_2c_2 + 5a_3c_1 + 8a_4c_0\right] \tag{1.77}$$

$$c_5 = \frac{1}{5a_0}\left[-2a_1c_4 + a_2c_3 + 4a_3c_2 + 7a_4c_1 + 10a_5c_0\right] \tag{1.78}$$

$$c_6 = \frac{1}{2a_0}\left[-a_1c_5 + a_3c_3 + 2a_4c_2 + 3a_5c_1 + 4a_6c_0\right] \tag{1.79}$$

$$c_7 = \frac{1}{7a_0}\left[-4a_1c_6 - a_2c_5 + 2a_3c_4 + 5a_4c_3 + 8a_5c_2 + 11a_6c_1 + 14a_7c_0\right] \tag{1.80}$$

$$c_8 = \frac{1}{8a_0}\left[-5a_1c_7 - 2a_2c_6 + a_3c_5 + 4a_4c_4 + 7a_5c_3 + 10a_6c_2 + 13a_7c_1 + 16a_8c_0\right] \tag{1.81}$$

In general, for $k > 0$,

$$c_k = \frac{1}{ka_0}\sum_{i=0}^{k-1}(2k - 3i)a_{k-i}c_i \tag{1.82}$$

1.8 Reciprocal of Square

The reciprocal of the square of a series is found by substituting $n = -2$ into this result:

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{-2} = \sum_{k=0}^{\infty} c_k x^k \tag{1.83}$$

The coefficients c_k are

$$\begin{aligned}
c_0 &= \frac{1}{a_0^2} && (1.84)\\
c_1 &= \frac{-2a_1c_0}{a_0} && (1.85)\\
c_2 &= \frac{1}{2a_0}\Big[-3a_1c_1 - 4a_2c_0\Big] && (1.86)\\
c_3 &= \frac{1}{3a_0}\Big[-4a_1c_2 - 5a_2c_1 - 6a_3c_0\Big] && (1.87)\\
c_4 &= \frac{1}{4a_0}\Big[-5a_1c_3 - 6a_2c_2 - 7a_3c_1 - 8a_4c_0\Big] && (1.88)\\
c_5 &= \frac{1}{5a_0}\Big[-6a_1c_4 - 7a_2c_3 - 8a_3c_2 - 9a_4c_1 - 10a_5c_0\Big] && (1.89)\\
c_6 &= \frac{1}{6a_0}\Big[-7a_1c_5 - 8a_2c_4 - 9a_3c_3 - 10a_4c_2 - 11a_5c_1 - 12a_6c_0\Big] && (1.90)\\
c_7 &= \frac{1}{7a_0}\Big[-8a_1c_6 - 9a_2c_5 - 10a_3c_4 - 11a_4c_3 - 12a_5c_2 - 13a_6c_1 - 14a_7c_0\Big] && (1.91)\\
c_8 &= \frac{1}{8a_0}\Big[-9a_1c_7 - 10a_2c_6 - 11a_3c_5 - 12a_4c_4 - 13a_5c_3 - 14a_6c_2\\
&\quad - 15a_7c_1 - 16a_8c_0\Big] && (1.92)
\end{aligned}$$

In general, for $k > 0$,

$$c_k = -\frac{1}{ka_0}\sum_{i=0}^{k-1}(2k-i)a_{k-i}c_i \tag{1.93}$$

1.9 Cube

The cube of a series is found by substituting $n = 3$ into this result:

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^3 = \sum_{k=0}^{\infty} c_k x^k \tag{1.94}$$

The coefficients c_k are

$$\begin{aligned}
c_0 &= a_0^3 && (1.95)\\
c_1 &= \frac{3a_1c_0}{a_0} && (1.96)\\
c_2 &= \frac{1}{a_0}\Big[a_1c_1 + 3a_2c_0\Big] && (1.97)\\
c_3 &= \frac{1}{3a_0}\Big[a_1c_2 + 5a_2c_1 + 9a_3c_0\Big] && (1.98)\\
c_4 &= \frac{1}{a_0}\Big[a_2c_2 + 2a_3c_1 + 3a_4c_0\Big] && (1.99)\\
c_5 &= \frac{1}{5a_0}\Big[-a_1c_4 + 3a_2c_3 + 7a_3c_2 + 11a_4c_1 + 15a_5c_0\Big] && (1.100)\\
c_6 &= \frac{1}{3a_0}\Big[-a_1c_5 + a_2c_4 + 3a_3c_3 + 5a_4c_2 + 7a_5c_1 + 9a_6c_0\Big] && (1.101)\\
c_7 &= \frac{1}{7a_0}\Big[-3a_1c_6 + a_2c_5 + 5a_3c_4 + 9a_4c_3 + 13a_5c_2 + 17a_6c_1 + 21a_7c_0\Big] && (1.102)\\
c_8 &= \frac{1}{2a_0}\Big[-a_1c_7 + a_3c_5 + 2a_4c_4 + 3a_5c_3 + 4a_6c_2 + 5a_7c_1 + 6a_8c_0\Big] && (1.103)
\end{aligned}$$

In general, for $k > 0$,

$$c_k = \frac{1}{ka_0}\sum_{i=0}^{k-1}(3k - 4i)a_{k-i}c_i \tag{1.104}$$

1.10 Reciprocal of Cube

The reciprocal of the cube of a series is found by substituting $n = -3$ into this result:

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{-3} = \sum_{k=0}^{\infty} c_k x^k \tag{1.105}$$

The coefficients c_k are

$$\begin{aligned}
c_0 &= \frac{1}{a_0^3} && (1.106)\\
c_1 &= \frac{-3a_1c_0}{a_0} && (1.107)\\
c_2 &= \frac{1}{a_0}\Big[-2a_1c_1 - 3a_2c_0\Big] && (1.108)\\
c_3 &= \frac{1}{3a_0}\Big[-5a_1c_2 - 7a_2c_1 - 9a_3c_0\Big] && (1.109)\\
c_4 &= \frac{1}{2a_0}\Big[-3a_1c_3 - 4a_2c_2 - 5a_3c_1 - 6a_4c_0\Big] && (1.110)\\
c_5 &= \frac{1}{5a_0}\Big[-7a_1c_4 - 9a_2c_3 - 11a_3c_2 - 13a_4c_1 - 15a_5c_0\Big] && (1.111)\\
c_6 &= \frac{1}{3a_0}\Big[-4a_1c_5 - 5a_2c_4 - 6a_3c_3 - 7a_4c_2 - 8a_5c_1 - 9a_6c_0\Big] && (1.112)\\
c_7 &= \frac{1}{7a_0}\Big[-9a_1c_6 - 11a_2c_5 - 13a_3c_4 - 15a_4c_3 - 17a_5c_2 - 19a_6c_1 - 21a_7c_0\Big] && (1.113)\\
c_8 &= \frac{1}{4a_0}\Big[-5a_1c_7 - 6a_2c_6 - 7a_3c_5 - 8a_4c_4 - 9a_5c_3 - 10a_6c_2 - 11a_7c_1 - 12a_8c_0\Big] && (1.114)
\end{aligned}$$

In general, for $k > 0$,

$$c_k = -\frac{1}{ka_0}\sum_{i=0}^{k-1}(3k - 2i)a_{k-i}c_i \tag{1.115}$$

1.11 Square Root

The square root of a series is found by substituting $n = 1/2$ into the previous result:

$$\sqrt{\sum_{k=0}^{\infty} a_k x^k} = \sum_{k=0}^{\infty} c_k x^k \tag{1.116}$$

The coefficients c_k are

$$\begin{aligned}
c_0 &= \sqrt{a_0} && (1.117)\\
c_1 &= \frac{a_1 c_0}{2a_0} && (1.118)\\
c_2 &= \frac{1}{4a_0}\Big[-a_1c_1 + 2a_2c_0\Big] && (1.119)\\
c_3 &= \frac{1}{2a_0}\Big[-a_1c_2 + a_3c_0\Big] && (1.120)\\
c_4 &= \frac{1}{8a_0}\Big[-5a_1c_3 - 2a_2c_2 + a_3c_1 + 4a_4c_0\Big] && (1.121)\\
c_5 &= \frac{1}{10a_0}\Big[-7a_1c_4 - 4a_2c_3 - a_3c_2 + 2a_4c_1 + 5a_5c_0\Big] && (1.122)\\
c_6 &= \frac{1}{4a_0}\Big[-3a_1c_5 - 2a_2c_4 - a_3c_3 + a_5c_1 + 2a_6c_0\Big] && (1.123)\\
c_7 &= \frac{1}{14a_0}\Big[-11a_1c_6 - 8a_2c_5 - 5a_3c_4 - 2a_4c_3 + a_5c_2 + 4a_6c_1 + 7a_7c_0\Big] && (1.124)\\
c_8 &= \frac{1}{16a_0}\Big[-13a_1c_7 - 10a_2c_6 - 7a_3c_5 - 4a_4c_4 - a_5c_3 + 2a_6c_2 + 5a_7c_1 + 8a_8c_0\Big] && (1.125)
\end{aligned}$$

In general, for $k > 0$,

$$c_k = \frac{1}{2ka_0}\sum_{i=0}^{k-1}(k-3i)a_{k-i}c_i \tag{1.126}$$

1.12 Reciprocal of Square Root

The reciprocal of the square root of a series is found by substituting $n = -1/2$ into the previous result:

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} c_k x^k \tag{1.127}$$

The coefficients c_k are

$$\begin{aligned}
c_0 &= \frac{1}{\sqrt{a_0}} && (1.128)\\
c_1 &= \frac{-a_1 c_0}{2a_0} && (1.129)\\
c_2 &= \frac{1}{4a_0}\Big[-3a_1c_1 - 2a_2c_0\Big] && (1.130)\\
c_3 &= \frac{1}{6a_0}\Big[-5a_1c_2 - 4a_2c_1 - 3a_3c_0\Big] && (1.131)\\
c_4 &= \frac{1}{8a_0}\Big[-7a_1c_3 - 6a_2c_2 - 5a_3c_1 - 4a_4c_0\Big] && (1.132)\\
c_5 &= \frac{1}{10a_0}\Big[-9a_1c_4 - 8a_2c_3 - 7a_3c_2 - 6a_4c_1 - 5a_5c_0\Big] && (1.133)\\
c_6 &= \frac{1}{12a_0}\Big[-11a_1c_5 - 10a_2c_4 - 9a_3c_3 - 8a_4c_2 - 7a_5c_1 - 6a_6c_0\Big] && (1.134)\\
c_7 &= \frac{1}{14a_0}\Big[-13a_1c_6 - 12a_2c_5 - 11a_3c_4 - 10a_4c_3 - 9a_5c_2 - 8a_6c_1 - 7a_7c_0\Big] && (1.135)\\
c_8 &= \frac{1}{16a_0}\Big[-15a_1c_7 - 14a_2c_6 - 13a_3c_5 - 12a_4c_4 - 11a_5c_3 - 10a_6c_2\\
&\quad -9a_7c_1 - 8a_8c_0\Big] && (1.136)
\end{aligned}$$

In general, for $k > 0$,

$$c_k = -\frac{1}{2ka_0}\sum_{i=0}^{k-1}(k+i)a_{k-i}c_i \tag{1.137}$$

1.13 Cube Root

The cube root of a series is found by substituting $n = 1/3$ into the previous result:

$$\sqrt[3]{\sum_{k=0}^{\infty} a_k x^k} = \sum_{k=0}^{\infty} c_k x^k \tag{1.138}$$

The coefficients c_k are

$$c_0 = \sqrt[3]{a_0} \tag{1.139}$$
$$c_1 = \frac{a_1 c_0}{3a_0} \tag{1.140}$$
$$c_2 = \frac{1}{3a_0}\Big[-a_1c_1 + a_2c_0\Big] \tag{1.141}$$
$$c_3 = \frac{1}{9a_0}\Big[-5a_1c_2 - a_2c_1 + 3a_3c_0\Big] \tag{1.142}$$
$$c_4 = \frac{1}{3a_0}\Big[-2a_1c_3 - a_2c_2 + a_4c_0\Big] \tag{1.143}$$
$$c_5 = \frac{1}{15a_0}\Big[-11a_1c_4 - 7a_2c_3 - 3a_3c_2 + a_4c_1 + 5a_5c_0\Big] \tag{1.144}$$
$$c_6 = \frac{1}{9a_0}\Big[-7a_1c_5 - 5a_2c_4 - 3a_3c_3 - a_4c_2 + a_5c_1 + 3a_6c_0\Big] \tag{1.145}$$
$$c_7 = \frac{1}{21a_0}\Big[-17a_1c_6 - 13a_2c_5 - 9a_3c_4 - 5a_4c_3 - a_5c_2 + 3a_6c_1 + 7a_7c_0\Big] \tag{1.146}$$
$$c_8 = \frac{1}{6a_0}\Big[-5a_1c_7 - 4a_2c_6 - 3a_3c_5 - 2a_4c_4 - a_5c_3 + a_7c_1 + 2a_8c_0\Big] \tag{1.147}$$

In general, for $k > 0$,

$$c_k = \frac{1}{3ka_0}\sum_{i=0}^{k-1}(k-4i)a_{k-i}c_i \tag{1.148}$$

1.14 Reciprocal of Cube Root

The reciprocal of the cube root of a series is found by substituting $n = -1/3$ into the previous result:

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{-\frac{1}{3}} = \sum_{k=0}^{\infty} c_k x^k \tag{1.149}$$

The coefficients c_k are

$$c_0 = \frac{1}{\sqrt[3]{a_0}} \tag{1.150}$$

$$c_1 = \frac{-a_1 c_0}{3a_0} \tag{1.151}$$

$$c_2 = \frac{1}{3a_0}\left[-2a_1c_1 - a_2c_0\right] \tag{1.152}$$

$$c_3 = \frac{1}{9a_0}\left[-7a_1c_2 - 5a_2c_1 - 3a_3c_0\right] \tag{1.153}$$

$$c_4 = \frac{1}{6a_0}\left[-5a_1c_3 - 4a_2c_2 - 3a_3c_1 - 2a_4c_0\right] \tag{1.154}$$

$$c_5 = \frac{1}{15a_0}\left[-13a_1c_4 - 11a_2c_3 - 9a_3c_2 - 7a_4c_1 - 5a_5c_0\right] \tag{1.155}$$

$$c_6 = \frac{1}{9a_0}\left[-8a_1c_5 - 7a_2c_4 - 6a_3c_3 - 5a_4c_2 - 4a_5c_1 - 3a_6c_0\right] \tag{1.156}$$

$$c_7 = \frac{1}{21a_0}\left[-19a_1c_6 - 17a_2c_5 - 15a_3c_4 - 13a_4c_3 - 11a_5c_2 - 9a_6c_1 - 7a_7c_0\right] \tag{1.157}$$

$$c_8 = \frac{1}{12a_0}\left[-11a_1c_7 - 10a_2c_6 - 9a_3c_5 - 8a_4c_4 - 7a_5c_3 - 6a_6c_2 - 5a_7c_1 - 4a_8c_0\right] \tag{1.158}$$

In general, for $k > 0$,

$$c_k = -\frac{1}{3ka_0}\sum_{i=0}^{k-1}(k+2i)a_{k-i}c_i \tag{1.159}$$

1.15 Identity

Series coefficients may be expressed by an identity transformation by setting $n = 1$. This allows each coefficient of a series to be expressed in terms of previous coefficients:

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} c_k x^k \tag{1.160}$$

The coefficients c_k are

$$c_0 = a_0 \tag{1.161}$$

$$c_1 = \frac{a_1 c_0}{a_0} = a_1 \tag{1.162}$$

$$c_2 = \frac{a_2 c_0}{a_0} = a_2 \tag{1.163}$$

$$c_3 = \frac{1}{3a_0}\Big[-a_1c_2 + a_2c_1 + 3a_3c_0\Big] = a_3 \tag{1.164}$$

$$c_4 = \frac{1}{2a_0}\Big[-a_1c_3 + a_3c_1 + 2a_4c_0\Big] = a_4 \tag{1.165}$$

$$c_5 = \frac{1}{5a_0}\Big[-3a_1c_4 - a_2c_3 + a_3c_2 + 3a_4c_1 + 5a_5c_0\Big] = a_5 \tag{1.166}$$

$$c_6 = \frac{1}{3a_0}\Big[-2a_1c_5 - a_2c_4 + a_4c_2 + 2a_5c_1 + 3a_6c_0\Big] = a_6 \tag{1.167}$$

$$c_7 = \frac{1}{7a_0}\Big[-5a_1c_6 - 3a_2c_5 - a_3c_4 + a_4c_3 + 3a_5c_2 + 5a_6c_1 + 7a_7c_0\Big] = a_7 \tag{1.168}$$

$$c_8 = \frac{1}{4a_0}\Big[-3a_1c_7 - 2a_2c_6 - a_3c_5 + a_5c_3 + 2a_6c_2 + 3a_7c_1 + 4a_8c_0\Big] = a_8 \tag{1.169}$$

In general, for $k > 0$,

$$c_k = \frac{1}{ka_0}\sum_{i=0}^{k-1}(k-2i)a_{k-i}c_i = a_k \tag{1.170}$$

Chapter 2

Reversion of Series

Suppose we are given a power series

$$y = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \tag{2.1}$$

This series may be solved for x as a power series in y:

$$x = A_1 y + A_2 y^2 + A_3 y^3 + A_4 y^4 + \cdots \tag{2.2}$$

The coefficients A_k are given by ($a_1 \neq 0$)

$$A_1 = \frac{1}{a_1} \tag{2.3}$$

$$A_2 = -\frac{a_2}{a_1^3} \tag{2.4}$$

$$A_3 = \frac{1}{a_1^5}\left[2a_2^2 - a_1 a_3\right] \tag{2.5}$$

$$A_4 = \frac{1}{a_1^7}\left[5a_1 a_2 a_3 - a_1^2 a_4 - 5a_2^3\right] \tag{2.6}$$

$$A_5 = \frac{1}{a_1^9}\left[6a_1^2 a_2 a_4 + 3a_1^2 a_3^2 + 14a_2^4 - a_1^3 a_5 - 21a_1 a_2^2 a_3\right] \tag{2.7}$$

$$A_6 = \frac{1}{a_1^{11}}\left[7a_1^3 a_2 a_5 + 7a_1^3 a_3 a_4 + 84a_1 a_2^3 a_3 - a_1^4 a_6 - 28a_1^2 a_2^2 a_4 - 28a_1^2 a_2 a_3^2 - 42a_2^5\right] \tag{2.8}$$

$$\begin{aligned} A_7 = \frac{1}{a_1^{13}}\Big[&8a_1^4 a_2 a_6 + 8a_1^4 a_3 a_5 + 4a_1^4 a_4^2 + 120a_1^2 a_2^3 a_4 + 180a_1^2 a_2^2 a_3^2 + 132a_2^6 - a_1^5 a_7 \\ &-36a_1^3 a_2^2 a_5 - 72a_1^3 a_2 a_3 a_4 - 12a_1^3 a_3^3 - 330a_1 a_2^4 a_3\Big] \end{aligned} \tag{2.9}$$