Numerical Table of Clebsch-Gordan Coefficients

Benjamin E. Chi

1962

1 Definition and Typical Applications

The Clebsch-Gordan coefficient (see part 4 for alternate names and notations) arises perhaps most fundamentally in the coupling together of two systems with angular momentum $\bar{j}_1$ and $\bar{j}_2$ respectively to form a composite system of angular momentum

$$\bar{j} = \bar{j}_1 + \bar{j}_2. \tag{1}$$

The axial components j_{1z} and j_{2z} of the two systems in the uncoupled representation are observables. Thus a complete set of commuting dynamical variables for the uncoupled representation is $\bar{j}_1 \cdot \bar{j}_1$, $\bar{j}_2 \cdot \bar{j}_2$, j_{1z}, j_{1z}, with corresponding eigenvalues $j_1(j_1+1)$, $j_2(j_2+1)$, m_1, m_2.

In the coupled representation, neither j_{1z} nor j_{2z} commute with $\bar{j} \cdot \bar{j}$ although their sum, $j_z = j_{1z} + j_{2z}$, does. So a complete set of observables for this case is $\bar{j} \cdot \bar{j}$, $\bar{j}_1 \cdot \bar{j}_1$, $\bar{j}_2 \cdot \bar{j}_2$, and j_z with respective eigenvalues $j(j+1)$, $j_1(j_1+1)$, $j_2(j_2+1)$, and m.

An eigenstate $|j\ j_1\ j_2\ m\rangle$ in the coupled representation must be a linear combination of product eigenstates $|j_1\ m_1\rangle\ |j_2\ m_2\rangle$ of the uncoupled representation, subject to the proviso that

$$m_1 + m_2 = m \tag{2}$$

since (1) is a vector equation implying equality of each of the three components. Of course, the magnitude j must lie between $j_1 + j_2$ and $|j_1 - j_2|$ since the three vectors form the sides of a triangle. This latter stipulation is sometimes referred to[(1)] by the symbol $\triangle(j_1\ j_2\ j)$.

This linear combination, then, may be written as

$$|j\ j_1\ j_2\ m\rangle = \sum_{m_1, m_2} C(j_1\ j_2\ j;\ m_1\ m_2\ m)\ |j_1\ m_1\rangle\ |j_2\ m_2\rangle. \tag{3}$$

The amplitudes $C(j_1\ j_2\ j;\ m_1\ m_2\ m)$ are called the Clebsch-Gordan coefficients. They vanish unless the following conditions are met:

$$\triangle(j_1\ j_2\ j),\ \ m_1 + m_2 = m,\ \ |m_1| \le j_1,\ \ |m_2| \le j_2,\ \ |m| \le j. \tag{4}$$

A very common application arises in the integration of products of spherical harmonics[1,2] $Y_{\ell m}$. Specifically,

$$\oiint Y^*_{\ell' m'}\, Y_{LM}\, Y_{\ell m}\, d\Omega = \sqrt{\frac{(2\ell+1)(2L+1)}{4\pi(2\ell'+1)}}\, C(\ell\; L\; \ell';\; m\; M\; m')\, C(\ell\; L\; \ell';\; 0\; 0\; 0). \tag{5}$$

For any operator $F_{LM} = \alpha Y_{LM}$, then, its matrix element between states $|\ell\; m\rangle$ and $|\ell' m'\rangle$ (with ℓ, L, ℓ' integral) is

$$\langle \ell'\; m' \mid F_{LM}\; |\ell\; m\rangle = \alpha\sqrt{\frac{(2\ell+1)(2L+1)}{4\pi(2\ell'+1)}}\, C(\ell\; L\; \ell';\; m\; M\; m')\, C(\ell\; L\; \ell';\; 0\; 0\; 0). \tag{6}$$

For coupled states $|\ell'\; m'\rangle$ where $\bar{j} = \bar{\ell} + \bar{s}$ $(s = 1/2)$, the following relation can also be shown to be true:

$$\langle j'\; m' \mid F_{LM}\; |j\; m\rangle = \alpha\sqrt{\frac{(2j+1)(2L+1)}{4\pi(2j'+1)}}\, C(j\; L\; j';\; m\; M\; m')\, C(j\; L\; j';\; -\frac{1}{2}\; 0\; -\frac{1}{2}) \tag{7}$$

This bears a superficial similarity to (6) but applies to quite a different situtation.

2 Explict Formulas

An explicit formula for the Clebsch-Gordan coefficient has been derived by Racah[3]:

$$\begin{aligned} C(j_1\; j_2\; j_3;\; m_1\; m_2\; m_3) &= \delta_{m_1+m_2,m_3}\; \sqrt{(2j_3+1)/(j_1+j_2+j_3+1)!} \\ &\times \sqrt{(j_1+j_2-j_3)!\; (j_2+j_3-j_1)!\; (j_3+j_1-j_2)!} \\ &\times \sqrt{(j_1+m_1)!\; (j_1-m_1)!\; (j_2+m_2)!\; (j_2-m_2)!\; (j_3+m_3)!\; (j_3-m_3)!} \\ &\times \sum_k \{(-1)^k/[(j_1+j_2-j_3-k)!\; (j_3-j_1-m_2+k)! \\ &\times\; (j_3-j_2+m_1+k)!\; (j_1-m_1-k)!\; (j_2+m_2-k)!\; k!\;]\} \end{aligned} \tag{8}$$

Terms in the sum vanish unless all factorial arguments in the term are nonnegative. This limits the range of k to a finite set of values. (In practice, all terms but one or two vanish.)

An elegant derivation of this formula has been given by Sharp[5].

Other explicit formulas exist[1,4] but are of lower symmetry than (8).

If one of the arguments be fixed in value, however, the formula simplifies considerably. Along this line the following algebraic tables have been published:

$j_2 = 1/2,\; 1$: Rose[1], Wigner[6].
$j_2 = 1/2$ to 2: Condon and Shortley[2].
$j_2 = 5/2$: Saito and Morita[7], Melvin and Swamy[4].
$j_2 = 3$: Falkoff *et al.*[8].

$j_2 = 7/2$ to 5, and also
$j_1 + j_2 - |m_3| = 1$ to 3: Shimpuku[9].

Numerical tables also exist. These fall into two categories:

a) Square roots of rational numbers:
$j_1 = 1/2,\ j_2 = 1/2$ to $7/2$;
$j_1 = 1,\ j_2 = 1/2$ to $5/2$;
$j_1 = 3/2,\ j_2 = 3/2, 2$;
$j_1 = j_2 = 2$: Heine[10].

$j_1 = 5,\ j_2 = 1/2$ to 6;
$j_1 = 11/2,\ j_2 = 1/2$ to 6;
$j_1 = 6,\ j_2 = 1/2$ tp 6: Shimpuku[9].

b) Decimal numbers:
$j_3 = 1$ to $9/2$ (10 decimal places): Simon[11].
$j_1, j_3 = 1/2$ to 10, $j_2 = 1$ to 6 in integral steps
(7 decimal places): the present work.

3 Symmetry Relations and Other Properties

Using (8) the following relations can be derived:

$$\begin{aligned}
C(j_1\ j_2\ j_3;\ m_1\ m_2\ m_3) &= (-1)^{j_1+j_2-j_3}\ C(j_1\ j_2\ j_3;\ -m_1\ -m_2\ -m_3) && (9)\\
&= (-1)^{j_1+j_2-j_3}\ C(j_2\ j_1\ j_3; m_2\ m_1\ m_3) && (10)\\
&= (-1)^{j_1-m_1} \sqrt{\frac{2j_3+1}{2j_2+1}}\ C(j_1\ j_3\ j_2;\ m_1\ -m_3\ -m_2) && (11)\\
&= (-1)^{j_2+m_2} \sqrt{\frac{2j_3+1}{2j_1+1}}\ C(j_3\ j_2\ j_1;\ -m_3\ m_2\ -m_1) && (12)\\
&= (-1)^{j_1-m_1} \sqrt{\frac{2j_3+1}{2j_2+1}}\ C(j_3\ j_1\ j_2;\ m_3\ -m_1\ m_2) && (13)\\
&= (-1)^{j_2+m_2} \sqrt{\frac{2j_3+1}{2j_1+1}}\ C(j_2\ j_3\ j_1;\ -m_2\ m_3\ m_1) && (14)
\end{aligned}$$

Also, if ℓ_1, ℓ_2, ℓ_3 be integers,

$$C(\ell_1\ \ell_2\ \ell_3;\ 0\ 0\ 0) = 0 \qquad \text{unless} \quad \ell_1 + \ell_2 + \ell_3 \quad \text{be even} \tag{15}$$

and, finally,

$$C(j_1\ 0\ j_3;\ m_1\ 0\ m_3) = \delta_{j_1 j_3}\ \delta_{m_1 m_3}. \tag{16}$$

4 Alternative Names and Notations; Related Quantities

The Clebsch-Gordan coefficient is variously called the Wigner coefficient, the C-coefficient and the vector-coupling coefficient.

Some other notations for the coefficient are as follows:

Condon and Shorley[2]: $(j_1\ j_2\ j_3\ m_3 \mid j_1\ j_2\ m_1\ m_2)$, $(j_3\ m_3 \mid m_1\ m_2)$, $(j_1\ j_2\ j_3\ m_3 \mid j_1\ m_1\ j_2\ m_2)$, $(j_1\ m_1\ j_2\ m_2 \mid j_1\ j_2\ j_3\ m_3)$. All these are very common in current literature.

Rose[1]: $C(j_1\ j_2\ j_3;\ m_1\ m_2)$.

Landau and Lifshitz[12]: $C^{j_3}_{m_1 m_2}$.

Blatt and Weisskopf[13]: $C_{j_1 j_2}(j_3, m_3;\ m_1, m_2)$.

Wigner[6]: $s^{(j_1 j_2)}_{j_3 m_1 m_2}$.

Other notations can be found summarized in a table in Edmonds[14].

Related to the Clebsch-Gordan coefficient is Racah's V-coefficient

$$V(j_1\, j_2\, j_3;\, m_1\, m_2\, m_3) = (-1)^{j_3 - m_3}\,(2j_3{+}1)^{-1/2}\, C(j_1\, j_2\, j_3;\, m_1\, m_2\, m_3), \quad (17)$$

and Wigner's 3-j symbol[6,14]:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 - j_2 - m_3}\,(2j_3{+}1)^{-1/2}\, C(j_1\, j_2\, j_3;\, m_1\, m_2\, m_3). \quad (18)$$

5 Method of Calculation and Arrangement of Table.

The IBM-650 program used in computing this table was originally written as a subroutine for evaluating matrix elements that arise in electromagnetic transitions. The calculation is carried out using Racah's formula (8) and is discussed more fully in an Appendix.

Accuracy of the values is probably about one part in the last place ($\pm 10^{-7}$).

The tables were computed using this subroutine and a set of commands to increment the arguments and arrange the output format. *All those sets of arguments which do not fulfill the requirements (4) yield zero coefficients are are omitted from the table.*

The arrangement of the table is as follows:

- $j_2 = 1$ to 6 (in integral steps)
  - $j_1 = 1/2$ to 10
    - $j_3 = |j_1 - j_2|$ to $\min(j_1 + j_2, 10)$
      - $m_2 = 0$ or $1/2$ to j_2
        - $m_1 = -j_1$ to j_1.

The most indented argument varies the most rapidly.

At the top of each page j_2 is listed. A new page is begun if j_2 changes its value. In each column the current value of j_1 is also tabulated. Arguments j_3, m_1, m_2 are shown for each entry; m_3 is not shown, being in every case $m_1 + m_2$.

`C` is the coefficient corresponding to the arguments. It is to be noted that only values for $m_2 \geq 0$ are shown. For negative m_2, symmetry relation (9) may be used, while if it is desired that j_2 be half-integral, relation (10) may be used. In either case, the required change does not alter the magnitude of the coefficient; the sign changes if the sign under `S` is `-`, it remains unchanged if `S` is `+`. `S` is just the factor $(-1)^{j_1+j_2-j_3}$.

No values are given for $j_2 = 0$ since relation (16) can be used here.

References

1. M.E. Rose, *Elementary Theory of Angular Momentum* (John Wiley and Sons, 1955).

2. E.U. Condon and G.H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, 1935).

3. G. Racah, *Phys. Rev.*, **62**, 438 (1942).

4. M.A. Melvin and N.V.V.J. Swamy, *Phys. Rev.*, **107**, 186 (1957).

5. R.T. Sharp, *Am. J. Phys.*, **28**, 16 (1960).

6. E.P. Wiger, *Group Theory* (Academic Press, 1959).

7. R. Saito and M. Morita, *Prog. Theo. Phys.* (Japan) **13**, 540 (1955).

8. D.L. Falkoff *et al.*, *Can. J. Phys.*, **30**, 253 (1952).

9. T. Shimpuku, *Suppl. Prog. Theo. Phys.* (Japan) **13**, 1 (1960).

10. V. Heine, *Group Theory in Quatum Mechanics* (Pergamon Press, 1960).

11. A. Simon, *Numerical Table of Chebsch-Gordan Coefficients*, Oak Ridge National Laboratory Report ORNL-1718 (1954, unpublished).

12. L.D. Landau and E.M. Lifshitz, *Quantum Mechanics, Non-Relativisitic Theory* tranlated by J.B. Sykes and J.S. Bell (Addison-Wesley Publishing Company, 1958).

13. J.J. Blatt and V.F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, 1952).

14. A.R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, 1957).